A Characterization of Complete Bipartite RAC Graphs

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Abstract. We provide a characterization of the complete bipartite graphs that admit a straight-line drawing where the edges can only cross at 90 degrees.

1 Introduction

The study of Right Angle Crossing drawings (RAC drawings for short) has been initiated in [1], motivated by recent cognitive experiments showing that edge crossings do not inhibit human task performances if the edges cross at 90 degrees [3, 4].

A tight upper bound on the number of edges of any straight-line RAC drawing has been proven [1]. Namely, the following result holds.

Theorem 1. [1] Any straight-line RAC drawing with \( n \) vertices and \( m \) edges is such that \( m \leq 4n - 10 \). Also, there are infinitely many graphs with \( n \) vertices and \( 4n - 10 \) edges that admit a straight-line RAC drawing.

Since every planar drawing is trivially a RAC drawing, Theorem 1 naturally raises the question of characterizing those non-planar graphs that have at most \( 4n - 10 \) edges and that admit a straight-line RAC drawing. In this paper we answer the question for the complete bipartite graphs. Indeed, there are infinitely many complete bipartite non-planar graphs \( K_{i,j} \) (\( i \leq j \)) for which \( ij \leq 4(i + j) - 10 \) (for example, if \( i \leq 4 \) or if \( i = 5 \) and \( j < 10 \)). This paper proves the following.

Theorem 2. A complete bipartite graph \( K_{n_1,n_2} \) \( (n_1 \leq n_2) \) admits a straight-line RAC drawing if and only if either \( n_1 \leq 2 \), or \( n_1 \leq 3 \) and \( n_2 \leq 4 \).

2 Preliminaries

Since we focus on straight-line drawings, in the remainder we say that a graph is a RAC graph if it admits a straight-line RAC drawing. The following properties of straight-line RAC drawings are immediate (see also [1]).

Property 1. Let \( D \) be a straight-line RAC drawing and let \( e_1, e_2, e_3 \) be three edges of \( D \) such that \( e_3 \) crosses both \( e_1 \) and \( e_2 \). Then \( e_1 \) is parallel to \( e_2 \) (see Figure 1(a)).

Property 2. Let \( D \) be a straight-line RAC drawing and let \( u \) be a vertex of \( D \). There is no edge that crosses two edges incident to \( u \) (see Figure 1(b)).

Property 3. Let \( D \) be a straight-line RAC graph and let \( T = (a, b, c) \) be a triangle such that \( (a, b) \) and \( (a, c) \) are edges of \( D \). If \( u \) is a vertex outside \( T \), and \( v, w \) are vertices inside \( T \), then there cannot exist two edges \( (u, v) \) and \( (u, w) \) of \( D \) that cross \( (a, b) \) and \( (a, c) \), respectively.

Proof. Since \( (u, v) \) and \( (u, w) \) make right angle crossings with \( (a, b) \) and \( (a, c) \), the interior angle of \( T \) at \( a \) must be greater than \( \pi \), which is impossible (see, e.g., Figure 1(c)).
Fig. 1. Basic properties of straight-line RAC drawings: (a) Two edges that cross a common edge must be parallel. (b) There cannot be an edge that crosses two edges incident to the same vertex. (c) A vertex cannot be separated from more than one neighbor by a triangle.

3 Characterizing Complete Bipartite RAC Graphs

By Fary’s theorem every planar graph has a straight-line crossing-free representation [2]. Therefore we focus on non-planar complete bipartite graphs. We first show examples of complete bipartite graphs that satisfy the condition of Theorem 1 but do not have a RAC representation, and then characterize the complete bipartite RAC graphs. We begin by giving some more definitions and some technical lemmas.

Let $G$ be a complete bipartite RAC graph and let $D$ be a straight-line RAC drawing of $G$. Assume that the vertices of $G$ are colored black and white, according to its bipartition. If $(a, b)$ and $(c, d)$ are two edges of $D$ that cross each other, then we call the subdrawing of $D$ induced by $\{a, b, c, d\}$ a butterfly (see, e.g., Figure 2(a)). Of course, a butterfly always consists of two black vertices and of two white vertices.

Fig. 2. (a) A butterfly in a straight-line RAC drawing of a complete bipartite graph. (b) Illustration of Lemma 1. (c) The polyline $P_{ac}$ (in bold) defined by the two white vertices of a butterfly.

**Lemma 1.** Let $G$ be a complete bipartite RAC graph and let $D$ be a straight-line RAC drawing of $G$. If $D$ contains a butterfly, the edges of the butterfly cannot cross any other edge in $D$.

**Proof.** Let $\{a, b, c, d\}$ be the vertices of the butterfly in $D$, and suppose that the two crossing edges of the butterfly are $(a, b)$ and $(c, d)$ (like in Figure 2(a)). Denote by $p$ the crossing point of $(a, b)$ and $(c, d)$. Suppose that there exists in $D$ an edge $(u, v)$ that crosses one of the edges $(a, b), (b, c), (c, d),$ or $(d, a)$. Properties 1 and 2 imply that $(u, v)$ cannot cross two of these edges, and so either $u$ or $v$ must be in the interior of one of the triangles $(b, c, p)$ or $(d, a, p)$; say that $u$ is in the interior of $(b, c, p)$. Note that $u$ is adjacent to either $a$ or $d$, since $G$ is a complete bipartite graph. Thus an edge from $u$ must cross either $(c, d)$ or $(a, b)$, as in Figure 2(b). From Property 2, we have a contradiction. $\square$

Let $D$ be a straight-line drawing of a complete bipartite graph and suppose that $D$ contains a butterfly. Let $x, y$ be two vertices of the butterfly having the same color. Denote by $p$ the crossing point of the butterfly and let $\ell_{xy}$ be the line passing through $x$ and $y$, $\ell_x \subseteq \ell_{xy}$ the half-line departing from $x$, and $\ell_y \subseteq \ell_{xy}$ the half-line departing from $y$. We denote by $P_{xy}$ the polyline obtained by the union of $\ell_x, \ell_y$ and the segments $(x, p), (y, p)$ (see Figure 2(c)). We say
that $P_{xy}$ separates a vertex $u$ from a vertex $v$ in $D$ if moving along $P_{xy}$ in one of the two possible directions, $u$ is to the right of $P_{xy}$ and $v$ is to the left of $P_{xy}$.

**Lemma 2.** Let $G$ be a complete bipartite RAC graph and let $D$ be a straight-line RAC drawing of $G$ with a butterfly $B$. Then $B$ contains two vertices $x, y$ of the same color such that $P_{xy}$ separates the other two vertices $w, z$ of $B$ from all vertices of $D$ having the same color as $w$ and $z$.

**Proof.** Let $e_1$ and $e_2$ be the two non-crossing edges of $B$, and let $\ell_1$ and $\ell_2$ be the two lines that contain $e_1$ and $e_2$, respectively.

Assume first that $\ell_1$ and $\ell_2$ are not parallel (see, e.g., Figure 3(a)). In this case they cross in a point $q$. Let $x$ be the vertex of $e_1$ that is closest to $q$, and let $y$ be the vertex of $e_2$ that is closest to $q$. Notice that $x, y$ have the same color. Let $u$ be a vertex of the same color as $w$ and $z$ and assume by contradiction that $P_{xy}$ leaves $u$ to the same side as $w$ and $z$. Since $G$ is a complete bipartite graph, then $u$ must be connected to both $x$ and $y$. However, since $\ell_1$ and $\ell_2$ do not intersect in the plane side determined by $P_{xy}$ in which $u$ lies, then at least one of the two edges between $(u, x)$ and $(u, y)$ must cross an edge of the butterfly (see, e.g., Figure 3(b)), which contradicts Lemma 1. It follows that $u$ must lie on the side of $P_{xy}$ opposite to $w$ and $z$.

![Illustration of the proof of Lemma 2. (a) Choosing vertices $x, y$. (b) Vertex $u$ cannot lie in the same side of $P_{xy}$ as $w$ and $z$.](image)

If $\ell_1$ and $\ell_2$ are parallel then one can see with the same argument as before that the statement holds for any pair $\{x, y\}$ of vertices of the butterfly having the same color.

**Lemma 3.** The complete bipartite graph $K_{4,4}$ is not a RAC graph.

**Proof.** Assume by contradiction that $D$ is a straight-line RAC drawing of $K_{4,4}$ and assume that the vertices of the graph are colored white and black. We denote by $u_i$ the white vertices and by $v_i$ the black vertices ($i \in \{1, 2, 3, 4\}$). Since every straight-line drawing of a $K_{4,4}$ contains an edge crossing, then $D$ contains at least one butterfly. Let $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ be the two crossing edges of this butterfly. Without loss of generality, assume that Lemma 2 holds for $x = u_1$ and $y = u_2$. This implies that the other two black vertices $v_3$ and $v_4$ must lie in the side of $P_{u_1u_2}$ opposite to $v_1$ and $v_2$. Two cases are possible:

**Case 1:** Vertices $v_1, v_2, u_1, u_2$ induce a butterfly (see, e.g., Figure 4(a)). Without loss of generality, assume that $(u_1, v_4)$ and $(u_2, v_3)$ are the two crossing edges of this butterfly. Consider the subdrawing $D'$ of $D$ induced by $u_1, u_2, v_i$ ($i \in \{1, 2, 3, 4\}$) and the white vertex $u_3$. It is immediate to verify that $u_3$ cannot lie in any of the internal faces of $D'$, because otherwise at least one of the edges connecting $u_3$ to a black vertex would cross an edge of the butterfly, which contradicts Lemma 1. Therefore, $u_3$ must lie in the external face of $D'$. Since by Lemma 1 no edge incident to $u_3$ can cross an edge of $D'$, the only possibility is that $u_3$ is outside the quadrilateral $Q = (v_1, v_2, v_3, v_4)$ and its connections with the four vertices of $Q$ do not intersect any sides of $Q$ (except at its corners). This implies that $u_3$ and two black vertices form a triangle $T$, which contains the other two black vertices inside. Due to symmetry assume, without loss of generality, that $T = (u_3, v_3, v_4)$ and hence that $v_1, v_2$ are inside $T$ (see Figure 4(b)). Denote now by $D''$ the subdrawing of $D$ resulting from $D'$ plus $u_3$ and its incident edges. We want to show that $u_4$ cannot lie
anywhere to complete $D$. For the same reasons explained for vertex $u_3$, also vertex $u_4$ must lie outside $Q$, and the incident edges cannot intersect $Q$. This implies in particular that if $u_4$ lies in the external face of $D''$, then its connections to $v_1$ and $v_2$ cannot cross the side $(v_3, v_4)$ of the quadrilateral (which would cause a crossing with some edges of a butterfly). Hence, if $u_4$ is in the external face of $D''$, then the edges $(u_4, v_1)$ and $(u_4, v_2)$ would cross the edges $(u_3, v_3)$ and $(u_3, v_4)$, which contradicts Property 3. If $u_4$ lies in the triangle $(u_3, v_2, v_3)$ then edge $(u_4, v_4)$ either crosses $Q$ (and this cannot happen) or it crosses the two edges $(u_3, v_2)$ and $(u_3, v_1)$, by violating Property 2. Symmetrically, $u_4$ cannot lie in the triangle $(u_3, v_1, v_4)$. Thus, the only possibility is that $u_4$ lies in the triangle $(u_3, v_1, v_2)$; in this case the edges $(u_4, v_3)$ and $(u_4, v_4)$ would cross $(u_3, v_2)$ and $(u_3, v_1)$, respectively, because they cannot cross $Q$. However, since both crossings must form $\pi/2$ angles, this would imply that the angle at $u_3$ inside $T$ is larger than $\pi$, which is impossible. Thus $D$ does not exist.

![Fig. 4. Illustration of the proof of Lemma 3: (a)-(b) Case 1. (c)-(d) Case 2.](image)

**Case 2: Vertices $v_1, v_2, u_1, u_2$ do not induce a butterfly** (see, e.g., Figure 4(c)). Assume, without loss of generality that triangle $(v_3, u_1, u_2)$ is inside triangle $(u_4, v_1, u_2)$. Denote by $D'$ the subdrawing induced by vertices $u_1, u_2, v_i$ ($i \in \{1, 2, 3, 4\}$). Vertices $u_3$ and $u_4$ cannot lie in any internal face of $D'$, because otherwise at least one of their connections with a black vertex would cross an edge of the butterfly, so violating Lemma 1. Therefore both $u_3$ and $u_4$ are in the external face of $D'$. In particular, edge $(u_3, v_3)$ must cross either $(u_2, v_2)$ or $(u_1, v_4)$, otherwise Property 2 would be violated. Assume, without loss of generality, that $(u_3, v_3)$ crosses $(u_2, v_2)$ (see Figure 4(d)). This implies that $(u_3, v_2, u_1, v_4)$ form a quadrilateral with $v_1$ inside. From Property 2, edge $(u_4, v_3)$ must cross $(u_1, v_4)$. However, this would imply that edge $(u_4, v_3)$ crosses also $(u_1, v_3)$, so violating Property 2. Thus, $D$ does not exist. □

**Lemma 4.** The complete bipartite graph $K_{3,5}$ is not a RAC graph.

**Proof.** Assume by contradiction that $D$ is a straight-line RAC drawing of $K_{3,5}$, and suppose that the vertices of the graph are colored white and black. Denote by $u_i$ the three white vertices ($i \in \{1, 2, 3\}$) and by $v_i$ the five black vertices ($i \in \{1, 2, 3, 4, 5\}$). Since the crossing number of $K_{3,5}$ is four [5], then $D$ has at least four crossings and each crossing determines a butterfly. Since $D$ has only three white vertices, there must be two butterflies containing the same pair of white vertices, say $v_1$ and $v_2$. From Lemma 1, the edges of the two butterflies cannot cross each other and also the two butterflies cannot share some edges, because this would violate Property 2. The two butterflies are therefore arranged as in Figure 4(a).

With the same arguments used in the proof of Lemma 3, the remaining white vertex $u_3$ must be outside the quadrilateral $Q = (v_1, v_2, v_3, v_4)$ and its connections with the four vertices of $Q$ do not intersect any side of $Q$ (except at its corners). This implies that $u_3$ and two black vertices form a triangle $T$, which contains all the other vertices inside. Due to symmetry assume, without loss of generality, that $T = (u_3, v_3, v_4)$, as shown in Figure 4(b). We now look at the position of the remaining black vertex $v_5$. It cannot be inside $Q$, because otherwise some of its incident edges would cross some edges of the butterflies, so contradicting Lemma 1. For the same reasons, since $u_1$ and $u_2$ are both internal to $Q$, $v_5$ cannot be inside any of the triangles $(u_3, v_1, v_2)$, $(u_3, v_2, v_3)$ and $(u_3, v_1, v_4)$. It follows that
Fig. 5. A straight-line RAC drawing of $K_{3,4}$.

$v_5$ must be outside $T$. In this case however, none of the edges $(v_5, u_1)$ and $(v_5, u_2)$ can cross the segment $(v_3, v_4)$ of $T$, because otherwise it would cross either edge $(v_3, u_2)$ or edge $(v_4, u_1)$, so contradicting Lemma 1. Therefore, edges $(v_5, u_1)$ and $(v_5, u_2)$ would cross edges $(u_3, v_3)$ and $(u_3, v_4)$, but this contradicts Property 3. Thus $D$ does not exist.

\[ \square \]

\textbf{Theorem 2.} A complete bipartite graph $K_{n_1,n_2}$ ($n_1 \leq n_2$) admits a straight-line RAC drawing if and only if either $n_1 \leq 2$, or $n_1 \leq 3$ and $n_2 \leq 4$.

\textit{Proof.} Every $K_{2,n}$ is planar. $K_{3,4}$ has a straight-line RAC drawing as shown in Figure 5. Every bipartite graph containing either $K_{4,4}$ or $K_{4,5}$ as a subgraph does not have a straight-line RAC drawing by Lemmas 3 and 4. \[ \square \]

We recall that there are upper bounds on the edge density similar to that of Theorem 1 for RAC drawings with at most one or at most two bends per edge [1]. It would be interesting to extend Theorem 2 to those complete bipartite graphs having a RAC drawing with at most one or two bends per edge. Also, we find it interesting to characterize families of RAC drawable graphs other than the complete bipartite graphs.

\textbf{References}