Simultaneous Embedding of Outerplanar Graphs, Paths, and Cycles

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Abstract

Let $G_1$ and $G_2$ be two planar graphs having some vertices in common. A simultaneous embedding of $G_1$ and $G_2$ is a pair of crossing-free drawings of $G_1$ and $G_2$ such that each vertex in common is represented by the same point in both drawings. In this paper we show that an outerplanar graph and a simple path can be simultaneously embedded with fixed edges such that the edges in common are straight-line segments while the other edges of the outerplanar graph can have at most one bend per edge. We then exploit the technique for outerplanar graphs and paths to study simultaneous embeddings of other pairs of graphs. Namely, we study simultaneous embedding with fixed edges of: (i) two outerplanar graphs sharing a forest of paths and (ii) an outerplanar graph and a cycle.

1 Introduction

The problem of visualizing a network that dynamically evolves over time is one of the emerging challenges in the field of graph drawing. This problem arises for example in the analysis of communication networks, where it is of interest both visualizing the changes in the network topology and understanding the evolution of the routing dynamics over time. Another example is the visualization of huge graphs, such as the Web graph, that are too large to be effectively displayed in a computer screen. One possible approach is to use an incremental exploration technique: The graph is drawn on a huge canvas and it is browsed by moving a small visualization window on the canvas. At any moment, only a small portion of the graph can be seen and some vertices and edges disappear while new others appear as the window moves. This is equivalent to having a dynamically changing graph in the window [6, 14, 17].

There is general consensus that for an effective visualization of a dynamically evolving network it is essential to preserve the user’s “mental map”, which means that consecutive drawings of similar graphs should not differ significantly [10]. Recent research has focused on maintaining the mental map by fixing the position of those vertices that are shared by consecutive drawings. A simultaneous embedding of two planar graphs $G_1$ and $G_2$ is a pair of drawings of $G_1$ and $G_2$ such that each drawing is crossing-free and each vertex that belongs to $V(G_1) \cap V(G_2)$ is represented by the same point in both drawings. Figures 1(a) and 1(b) show two planar graph $G_1$ and $G_2$. Two crossing-free drawings of $G_1$ and $G_2$ that defines a simultaneous embedding of $G_1$ and $G_2$ are shown in Figures 1(c) and 1(d). The two drawings are depicted together in Figure 1(e).

Simultaneous embeddings have been first defined by Brass et al. [2] who present several pioneering results with the additional requirement that he edges are straight-line segments. Simultaneous embeddings with straight-line edges, also called geometric simultaneous embeddings, are used by Duncan, Eppstein, and Kobourov to show that all graph with vertex degree at most four have geometric thickness two [9]. Systems devoted to simultaneous drawings of graphs have also been implemented (see, e.g., [12, 16]).

Unfortunately, geometric simultaneous embeddings are not always realizable, even when the graphs to be displayed have a very simple structure: Erten and Kobourov [11] show that there exists a pair of planar graphs consisting of a triangulation and of a simple path for which a geometric simultaneous embedding cannot be realized. Eventually, Erten and Kobourov [11] show how to locate a nonrealizable pair of graphs.

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Figure 1: (a) A planar graph $G_1$. (b) A planar graph $G_2$. (c) A crossing-free drawing of $G_1$. (d) A crossing-free drawing of $G_2$. (e) A simultaneous embedding of $G_1$ and $G_2$. 

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is not possible. Motivated by this negative result Erten and Kobourov extend the definition of geometric simultaneous embeddings and allow bends along the edges. They distinguish between simultaneous embeddings with fixed edges where edges that are in common have the same drawing, and simultaneous embeddings where common edges share their endpoints but can have different representations. The two drawings shown in Figures 1(c) and 1(d) do not define a simultaneous geometric embedding of $G_1$ and $G_2$, since some edges are not drawn as straight-line segments. Neither they define a simultaneous embeddings with fixed edges because edge $(c, h)$ has two different representation in the two drawings. In this paper we revisit the elegant technique of Erten and Kobourov [11] and extend and generalize some of their results. Our main contribution is twofold.

- We show that an outerplanar graph and a simple path can be simultaneously embedded with fixed edges such that the edges of the path are straight-line segments while the edges of the outerplanar graph have at most one bend per edge. This extends a previous result on trees and simple paths [11].

- We exploit the technique for outerplanar graphs and paths to study simultaneous embeddings of other pairs of graphs. Namely, we study simultaneous embedding with fixed edges of: (i) two outerplanar graphs sharing a forest of paths and (ii) an outerplanar graph and a cycle. The result on pairs of outerplanar graphs sheds some light on an open question regarding simultaneous embedding of pairs of trees [11].

The remainder of this paper is organized as follows. Preliminaries are in Section 2. In Section 3 we revisit the technique in [11] for the computation of simultaneous embeddings. An additional contribution of Section 3 is that, based on a couple of observations that combine the technique of Erten and Kobourov with existing literature, we can improve some of the results in [11]. Namely, in Section 3 we show that: (i) two planar graphs have a simultaneous embedding with at most two bends per edge; (ii) two series-parallel graphs have a simultaneous embedding with at most one bend per edge. Algorithms for simultaneous embeddings with fixed edges are described in Section 4. In Section 4.1, consistently with the definitions given in the literature, we assume that the two graphs $G_1$ and $G_2$ have their vertex set in common, i.e. $V(G_1) = V(G_2)$. Finally, conclusions and open problems can be found in Section 5.

2 Preliminaries

We assume familiarity with basic graph theory definitions [13]. A drawing $\Gamma$ of a graph $G$ is a function that maps each vertex $v \in V(G)$ to a point $\Gamma(v)$ in the plane and each edge $(u, v) \in E(G)$ to a simple Jordan curve whose endpoints are $\Gamma(u)$ and $\Gamma(v)$. A simultaneous embedding [11] $\Psi = (\Gamma_1, \Gamma_2)$ of two planar graphs $G_1$ and $G_2$ such that $V(G_1) = V(G_2) = V$ is a pair of crossing-free drawings $\Gamma_1$ and $\Gamma_2$ of $G_1$ and $G_2$, respectively, such that for every vertex $v \in V$ we have $\Gamma_1(v) = \Gamma_2(v)$. If every edge $e \in E(G_1) \cap E(G_2)$ is represented with the same simple open Jordan curve both in $\Gamma_1$ and in $\Gamma_2$ we say that $\Psi$ is a simultaneous embedding of $G_1$ and $G_2$ with fixed edges [11]. If the edges of $G_1$ and $G_2$ are represented with straight-line segments in $\Gamma_1$ and $\Gamma_2$ we say that $\Psi$ is a simultaneous geometric embedding of $G_1$ and $G_2$ [11]. The existence of simultaneous geometric embeddings for pairs of simple paths, simple cycles, and caterpillars is shown in [2] where also counter-examples for pairs of general planar graphs, pairs of outerplanar graphs, and triples of simple paths are presented.

3 Simultaneous Embedding of Planar Graphs

The algorithms to compute a simultaneous embedding of a pair of planar graphs and of a pair of trees presented in [11] are an elegant combination of the technique by Kaufmann and Wiese [15] for point-set embedding and of the simultaneous embedding strategy for two simple paths by Brass et al. [2]. In this section we first recall the main ideas of Erten and Kobourov for simultaneous embedding of two planar
graphs $G_1$ and $G_2$ and then make a couple of observations on how to combine these ideas with known literature in order to extend some of the results in [11]. Suppose first that both $G_1$ and $G_2$ are Hamiltonian and let $C_1$ and $C_2$ be two Hamiltonian cycles of $G_1$ and $G_2$, respectively. The algorithm by Erten and Kobourov consists of three steps (refer to Figure 2).

**Step 1.** For each cycle, one arbitrarily chosen edge (called closing edge in the following) is removed in order to obtain two Hamiltonian paths $P_1$ and $P_2$ for $G_1$ and $G_2$, respectively. For example, Figures 2(a) and 2(b) show two Hamiltonian graphs $G_1$ and $G_2$; in the figures, the bold edges denote the two Hamiltonian cycles $C_1$ and $C_2$. The closing edge of $C_1$ is $(a, h)$, the closing edge of $C_2$ is $(d, g)$.

**Step 2.** A simultaneous geometric embedding of the two Hamiltonian paths $P_1$ and $P_2$ is computed by the algorithm of Brass et al. [2]. Namely, starting from one of its end-vertices, $P_1$ is traversed and an increasing positive integer is given to each visited vertex; starting from one of its end-vertices, $P_2$ is traversed and a second increasing positive integer is given to each visited vertex. Since there is a mapping between the vertices of $P_1$ and of $P_2$ we have that every vertex of $P_1$ (and of $P_2$) is now associated with a pair of numbers. For a vertex $v$, the first number (the one defined when traversing $P_1$) is the $x$-coordinate and the second number is the $y$-coordinate of the grid point representing $v$. The defined set of points supports a simultaneous geometric embedding of $P_1$ and of $P_2$ because each of the two paths is either $x$- or $y$-monotone and hence it is crossing-free. Figure 2(c) shows the simultaneous embedding of the two Hamiltonian paths $P_1$ and $P_2$ for the graphs of Figures 2(a) and 2(b). In what follows the $x$- and $y$-coordinates of $v$ will be denoted as $x(v)$ and $y(v)$, respectively.

**Step 3.** The remaining edges of $G_1$ and of $G_2$ are added to the drawing by using the technique of Kaufmann and Wiese [15]. Let $v_0, v_1, \ldots, v_{n-1}$ be the vertices of $G_1$ (and of $G_2$) in the order they appear along $P_1$. Let $\Delta x_1 = \min_i \{|x(v_{i+1}) - x(v_i)|\}$ and $\Delta y_1 = \max_i \{|y(v_{i+1}) - y(v_i)|\}$ (note that $|x(v_{i+1}) - x(v_i)| = 1$ for each $0 \leq i < n - 1$, and hence $\Delta x_1 = 1$). Let $\delta_1 = \Delta y_1 / \Delta x_1$, i.e. $\delta_1$ is the maximum slope of a segment in the drawing of $P_1$. The closing edge for cycle $C_1$ is drawn as a polyline with two segments whose slopes are $\delta_1'$ and $-\delta_1'$, where $\delta_1' = \delta_1 + \epsilon$ and $\epsilon$ is an arbitrary positive constant. See Figure 2(d) as an example of how the closing edge is added to the drawing. The remaining edges of $G_1$ are partitioned into those edges that are inside $C_1$ and those edges that are outside $C_1$. The edges that are inside (outside) $C_1$ are drawn inside (outside) $C_1$ as polylines each consisting of two segments having slopes $\delta_1'$ and $-\delta_1'$, respectively; see Figure 2(e). Possible overlaps between segments corresponding to different edges can be removed by a simple rotation technique described in [15]. The drawing of $G_2$ is computed with an analogous procedure. Let $u_0, u_1, \ldots, u_{n-1}$ be the vertices of $G_2$ (and of $G_1$) in the order they appear along $P_2$. Let $\Delta x_2 = \max_i \{|x(u_{i+1}) - x(u_i)|\}$ and $\Delta y_2 = \min_i \{|y(u_{i+1}) - y(u_i)|\}$ (note that $|y(u_{i+1}) - y(u_i)| = 1$ for each $0 \leq i < n - 1$, and hence $\Delta y_2 = 1$). Let $\delta_2 = \Delta y_2 / \Delta x_2$, i.e. $\delta_2$ is the minimum slope of a segment in the drawing of $P_2$ (informally speaking, $\delta_2$ is the maximum slope of a segment of $P_2$ “with respect to the $y$-axis”). The closing edge for cycle $C_2$ is drawn as a polyline with two segments whose slopes are $\delta_2'$ and $-\delta_2'$, where $\delta_2' = \delta_2 - \epsilon$ and $\epsilon$ is an arbitrary positive constant (see Figure 2(d)). Also in this case, the edges that are inside (outside) $C_2$ are drawn inside (outside) $C_2$ as polylines each consisting of two segments having slopes $\delta_2'$ and $-\delta_2'$, respectively (see Figure 2(e)). Possible overlaps between segments corresponding to different edges can be removed by the rotation technique described in [15].

Erten and Kobourov show that the size of the obtained drawing is $O(n^2) \times O(n^2)$. Indeed, the simultaneous geometric embedding of $P_1$ and $P_2$ (i.e. the result of Step 2) has size $n \times n$ since each vertex has a different $x$-coordinate in the range $[0, n - 1]$ and a different $y$-coordinate in the range $[0, n - 1]$. The area of the drawing of $G_1$ and $G_2$ (i.e. the result of Step 3) is larger because the bends of the edges of $G_1$ and $G_2$ lie outside of the $n \times n$ grid. We have that $\delta_1$ is at most $n$ since $\Delta x_1 = 1$ and $\Delta y_1 \leq n$. Therefore the slope of the segments representing the bent edges of $G_1$ is $\delta_1 = n + \epsilon$ and the maximum height of a bent edge of $G_1$ is $O(n^2)$. Analogously, $\delta_2$ is at least $1/n$ since $\Delta x_2 \leq n$ and $\Delta y_2 = 1$. Therefore the slope of the segments representing the bent edges of $G_2$ is $\delta_2 = 1/n - \epsilon$ and the maximum width of a bent edge of $G_2$ is $O(n^2)$ (see Figure 3). It follows that the overall area is $O(n^2) \times O(n^2)$. Concerning the time complexity, the drawing
Figure 2: (a) A planar graph $G_1$ with Hamiltonian cycle $C_1$. (b) A planar graph $G_2$ with Hamiltonian cycle $C_2$. (c) A simultaneous geometric embedding of $P_1$ and $P_2$. The closing edges chosen for $C_1$ and $C_2$ are $(a, h)$ and $(g, d)$, respectively. (d) A simultaneous embedding of $C_1$ and $C_2$. (e) A simultaneous embedding of $G_1$ and $G_2$. 
can be computed in $O(n)$ time if the two Hamiltonian cycles $C_1$ and $C_2$ are given. Indeed, Step 2 can be performed by traversing first $P_1$ and then $P_2$, which requires $O(n)$ time. The fact that the complexity of Step 3 is also $O(n)$ is a consequence of the drawing algorithm by Kaufmann and Wiese [15].

Figure 3: Maximum width of a bent edge.

**Lemma 1** [11] Let $G_1$ and $G_2$ be two Hamiltonian graphs such that $V(G_1) = V(G_2) = V$ and let a Hamiltonian cycle of $G_1$ and a Hamiltonian cycle of $G_2$ be given. $G_1$ and $G_2$ can be simultaneously embedded with fixed edges in $O(n)$ time, using at most one bend per edge, on an integer grid of size $O(n^2) \times O(n^2)$, where $n = |V|$.

Observe that if the edges shared by $G_1$ and $G_2$ all belong to paths $P_1$ and $P_2$, then the simultaneous embedding computed by the technique of Erten and Kobourov is a simultaneous embedding with fixed edges, since the edges of $P_1$ and $P_2$ are drawn as straight-line segments. Also, if $G_2$ is a simple path we have that $P_2$ coincides with $G_2$ and the area bound is $O(n^2) \times O(n)$ because $G_2$ has no bent edge. We can summarize this discussion as follows.

**Observation 1** Let $G_1$ and $G_2$ be two Hamiltonian graphs such that $V(G_1) = V(G_2) = V$. Let a Hamiltonian cycle $C_1$ of $G_1$ and a Hamiltonian cycle $C_2$ of $G_2$ be given and let $P_1$ and $P_2$ be two Hamiltonian paths obtained by removing a closing edge from $C_1$ and from $C_2$, respectively. If $E(G_1) \cap E(G_2) \subseteq E(P_1) \cap E(P_2)$ then the technique by Erten and Kobourov computes a simultaneous embedding of $G_1$ and $G_2$ with fixed edges.

**Observation 2** Let $G$ be a Hamiltonian graph and let $P$ be a simple path such that $V(G) = V(P) = V$ and let a Hamiltonian cycle of $G_1$ be given. The technique by Erten and Kobourov computes a simultaneous embedding of $G$ and $P$ in $O(n)$ time, using at most one bend per edge, on an integer grid of size $O(n^2) \times O(n)$, where $n = |V|$.

The statement of Lemma 1 can be extended also to the case that no Hamiltonian cycles are given as part of the input, provided that there exists an $O(n)$-time technique to compute such two cycles. A graph that can be augmented with only edge addition to become Hamiltonian is said to be sub-Hamiltonian. Although recognizing sub-Hamiltonian graphs is NP-hard in general, there are other families of graphs other than trees that are known to be sub-Hamiltonian and for which an augmented Hamiltonian cycle can be found in $O(n)$ time. Among such families, we mention here trees, outerplanar graphs and series-parallel graphs (see, e.g., [1, 5, 7]). What follows extends Theorem 3 of [11] that only considers trees.

**Theorem 1** Let $G_1$ and $G_2$ be two graphs such that $V(G_1) = V(G_2) = V$ and $G_i$ ($i = 1, 2$) is either a series-parallel graph, or an outerplanar graph, or a tree. $G_1$ and $G_2$ can be simultaneously embedded in $O(n)$ time, using at most one bend per edge, on an integer grid of size $O(n^2) \times O(n^2)$, where $n = |V|$.
Suppose now that $G_1$ and $G_2$ are not Hamiltonian. Erten and Kobourov rely on a $O(n)$-time algorithm by Kaufmann and Wiese [15] to augment $G_1$ and $G_2$ and make them 4-connected. The augmentation algorithm by Kaufmann and Wiese [15] (which is in turn based on a result by Chiba and Nishizeki [3]) augments the two graphs by adding dummy edges and by splitting each edge with at most one dummy vertex. By means of another result of Chiba and Nishizeki [4], it is possible to find a Hamiltonian cycle in the augmented graph in $O(n)$ time.

A simultaneous embedding of the augmented graphs can then be computed in linear time by using the technique of Lemma 1. After such an embedding is computed the dummy edges are removed and the dummy vertices are treated as bend points. As a result, every edge $(u, v)$ that is split by a dummy vertex $w$ ends up having at most three bends, one between $u$ and $w$, one at $w$ and one between $w$ and $v$. Observe that the bend at $w$ can be avoided if the two segments of $(u, w)$ and $(w, v)$ incident on $w$ have the same slope. In [15] it is described how to rotate the segments incident on $w$ so to avoid the third bend. However this rotation increases the area of the drawing, and Erten and Kobourov do not apply the rotation. We recall that in [8] it is presented a variant of the algorithm by Kaufmann and Wiese [15] such that for each dummy vertex $w$ that splits edge $(u, v)$, the two segments of $(u, w)$ and $(w, v)$ incident on $w$ have the same slope. Thus no rotation of the edges is required to avoid the third bend. The following improves Theorem 2 of [11].

**Theorem 2** Let $G_1$ and $G_2$ be two planar graphs such that $V(G_1) = V(G_2) = V$. $G_1$ and $G_2$ can be simultaneously embedded in $O(n)$ time, using at most two bend per edge, on an integer grid of size $O(n^2) \times O(n^2)$, where $n = |V|$.

### 4 Simultaneous Embedding with Fixed Edges

Concerning the pair $<\text{tree, path}>$, the following is known.

**Theorem 3** [11] Let $T$ be a tree and let $P$ be a simple path such that $V(T) = V(P) = V$. $T$ and $P$ can be simultaneously embedded with fixed edges in $O(n)$ time, using at most one bend for each edge of $T$ and zero bends for each edge of $P$, on an integer grid of size $O(n) \times O(n^2)$, where $n = |V|$.

The idea of Erten and Kobourov behind Theorem 3 exploits the technique described in Section 3 and can be described as follows. Tree $T$ is augmented to become a planar graph $G'$ such that $G'$ has a Hamiltonian cycle $C_1$ that contains all edges of $E(P) \cap E(T)$. Path $P$ is augmented to become a cycle $C_2$. Graphs $G'$ and $C_2$ are then simultaneously embedded by means of the technique of Section 3 and the dummy edges are removed. If in Step 1 the closing edges of $C_1$ and $C_2$ are chosen to be two edges that are not in $E(T) \cap E(P)$, then the drawing produced is a simultaneous embedding with fixed edges (Observation 1). The bound on the area follows from Observation 2.

It is worth remarking that the elegance of the proof of Erten and Kobourov for Theorem 3 is to reduce the tree-path simultaneous embedding problem with fixed edges to the combinatorial question of finding an augmented Hamiltonian cycle in the augmented graph $G'$ with the additional constraint that the cycle must contain all edges of $E(P) \cap E(T)$. In Subsection 4.1 we use the same approach of Erten and Kobourov to extend the result of Theorem 3. Namely, we will prove that an outerplanar graph $G$ and a simple path $P$ can be simultaneously embedded with fixed edges by showing that $G$ can be augmented to a graph $G'$ that is still planar and that has a Hamiltonian cycle containing all edges of $E(G) \cap E(P)$. The result of Subsection 4.1 is then extended and generalized in Subsection 4.2.

#### 4.1 Outerplanar Graphs and Paths

We need some more definitions. A linear ordering of an $n$-vertex graph $G$ is a bijection $\sigma : V(G) \rightarrow \{1, 2, \ldots, n\}$. We write $u <_\sigma v$ to mean $\sigma(u) < \sigma(v)$. In some cases it will be useful to express $\sigma$ by the list $v_0, v_1, \ldots, v_{n-1}$, where $v_i <_\sigma v_j$ if and only if $0 \leq i < j \leq n-1$. In what follows if no ambiguity arises about $\sigma$ we will write $u < v$ instead of $u <_\sigma v$. Let $G$ be a graph and let $\sigma$ be a linear ordering of $G$. Let $e_0 = (v_i, v_j)$ and $e_1 = (v_h, v_l)$ be two edges of $G$. Edges $e_0$ and $e_1$ cross if $i < h < j < l$ (i.e. if
v_i <_v v_h <_v v_j <_v v_1). Edges e_0 and e_1 nest if i < h < l < j (i.e. if v_i <_v v_h <_v v_1 <_v v_j); in this case we also say that e_1 is nested inside e_0.

A k-page book embedding \( \phi(G) \) consists of a linear ordering \( \sigma \) of \( G \) along with a partition \( \{ E_l | 0 \leq l \leq k-1 \} \) of \( E(G) \) such that no two edges in the same partition set \( E_l \) cross. Each set \( E_l \) is called a page. Let \( \phi(G) \) be a k-page book embedding of a graph \( G \). A \( r \)-rainbow in \( \phi(G) \) is a set of \( r \) edges \( e_0, e_1, \ldots, e_{r-1} \in E(G) \) such that they are in the same page and \( e_{i+1} \) is nested inside \( e_i \) (\( 0 \leq i < r-1 \)). Edges \( e_0 \) and \( e_1 \) are called the top edge and the second edge of the \( r \)-rainbow, respectively. Bernhart and Kainen [1] prove the following lemma that will be useful in what follows.

**Lemma 2** [1] Let \( \phi(G) \) be a k-page book embedding of a graph \( G \) with linear ordering \( \sigma \) such that \( v_0 <_v v_1 <_v \cdots <_v v_{n-1} \). There exists a k-page book embedding \( \phi'(G) \) of \( G \) with total ordering \( \sigma' \) such that \( v_1 <_v v_2 <_v \cdots <_v v_{n-1} <_v v_0 \).

The minimum value \( k \) such that \( G \) admits a k-page book embedding is called the page number of \( G \). A graph has page number one if and only if it is outerplanar and a graph has page number two if and only if it is sub-Hamiltonian [1]. In what follows we are interested to 1- and 2-page book embedding. We recall that a graph has page number one if and only if it is outerplanar and a graph has page number two if and only if it is sub-Hamiltonian [1]. In what follows we are interested to 1- and 2-page book embedding. We recall that a graph has page number one if and only if it is outerplanar and a graph has page number two if and only if it is sub-Hamiltonian.

Furthermore \( G' \) admits a 2-page book embedding \( \phi(G') \) such that:

- \( \phi(G') \) has the same linear ordering \( \sigma \) as \( \phi(G) \);
- all edges of \( E(G') \cap E(G) \) are in page \( E_0 \) of \( \phi(G') \) and all edges of \( E(G') \setminus E(G) \) are in page \( E_1 \) of \( \phi(G') \);
- for each edge \((v_g, v_h) \in E(G') \setminus E(G)\) we have \( i \leq g < h \leq j+1 \).

**Proof:** Let \( G^* \) be the graph obtained from \( G \) by removing all the edges of \( E(G) \setminus E' \). Let \( \phi(G') \) be a 1-page book embedding of \( G^* \) with the same linear ordering as \( \phi(G) \). For each edge \( e \in E' \), the weight of \( e \) is the maximum \( r \) such that there exists a \( r \)-rainbow in \( \phi(G') \) having \( e \) as its top edge; if \( e \) is not the top edge of any rainbow, the weight of \( e \) is 0.

We prove the statement by induction on the weight \( w \) of \( e^* \). If \( w = 0 \), i.e. no edge of \( E' \) is nested inside \( e^* \), we choose \( \pi \) as follows. If there is no vertex between \( v_i \) and \( v_j \) along the spine, we choose \( \pi = v_i, v_j, v_{j+1} \) (see also Figure 4(a)); otherwise, \( \pi \) is obtained by concatenating edge \((v_i, v_j)\) with the simple path \( v_i, v_{j-1}, \ldots, v_1 \) and with edge \((v_{i+1}, v_{j+1})\) (see also Figure 4(b)). The edges of \( \pi \) that are not in \( G \) are augmenting edges. Let \( G' \) be the resulting augmented graph. We partition the edges of \( G' \) into two pages: \( E_0 \) contains the edges of \( E(G') \cap E(G) \), and \( E_1 \) contains the edges of \( E(G') \setminus E(G) \). We prove that the linear ordering \( \sigma \) (which is a linear ordering of \( G' \) since \( V(G') = V(G) \)) along with the partition of \( E(G') \) into the two pages \( E_0 \) and \( E_1 \) is a 2-page book embedding.
The edges of $E_0$ do not cross each other because they are all edges of $G$ and do not cross in $\phi(G)$. All edges in $E_1$ connect pairs of vertices that are consecutive in $\sigma$ except, possibly, edge $(v_{i+1}, v_{j+1})$. It follows that there cannot be any crossing in $E_1$.

Suppose now that $w = k \geq 1$ and that the statement is true for $w < k$. Let $e_0 = (v_{i_0}, v_{h_0}), e_1 = (v_{i_1}, v_{j_1}), \ldots, e_h = (v_{h_{h-1}}, v_{h-1})$ be the second edges of the rainbows of $\phi(G^*)$ that have $e$ as their top edge (see Figure 5). The weight of each edge $e_l$ ($0 \leq l \leq h - 1$) is at most $k - 1$ and by inductive hypothesis there exists a simple path $\pi_l$ from $v_{i_l}$ to $v_{h_{l+1}}$ that contains $e_l$, all edges of $E'$ that are nested inside $e_l$, and all edges between vertices that are non-consecutive along the spine and that do not cross in $\phi(G)$ (Figure 5). The edges of $\pi$ that are not in $G$ are augmenting edges. Path $\pi$ starts at $v_{i_l}$ and ends at $v_{j_{l+1}}$, it contains $e^*$, and by induction it contains all vertices between $v_i$ and $v_{j+1}$ and all edges of $E'$ nested inside $e$.

Let $G'$ be the augmented graph obtained by the edge addition described above. We partition the edges of $G'$ into two pages: $E_0$ contains the edges of $E(G') \cap E(G)$, and $E_1$ contains the edges of $E(G') \setminus E(G)$. We prove that the linear ordering $\sigma$ (which is a linear ordering of $G'$ since $V(G') = V(G)$) along with the partition of $E(G')$ into the two pages $E_0$ and $E_1$ is a 2-page book embedding.

The edges in $E_0$ do not cross each other because they are all edges of $G$ and do not cross in $\phi(G)$. Let $d_0 = (v_{g_0}, v_{h_0})$ and $d_1 = (v_{g_1}, v_{h_1})$ be two edges in $E_1$ both connecting two vertices that are non-consecutive in $\sigma$. If $d_0$ and $d_1$ are both edges of a simple path $\pi_l$ ($0 \leq l \leq h - 1$), then they do not cross by induction. If $d_0$ and $d_1$ are edges of two different simple paths $\pi_{i_0}$ and $\pi_{i_1}$ ($0 \leq l_0 < l_1 \leq h - 1$), then by induction we have that $i_0 < j_0 < h_0 < j_{i_0} + 1$ and $i_1 < g_1 < h_1 < j_{i_1} + 1$. Since $j_{i_0} + 1 \leq i_1$, then $d_0$ and $d_1$ do not cross. The only edge in $E_1$ that connects vertices that are non-consecutive along the spine and that does not belong to any $\pi_l$ ($0 \leq l \leq h - 1$) is $(v_{i+1}, v_{j+1})$. Let $d_0$ be the edge $(v_{i+1}, v_{j+1})$ and let $d_1$ be an edge of a path $\pi_l$ ($0 \leq l \leq h - 1$); we have $i + 1 < i_l < j_l < j + 1$ and by induction $i_l < g_l < h_l < j_l + 1$ which implies that the two edges do not cross. Clearly the described recursive strategy can be performed in $O([j - i])$ time.
by scanning all the vertices between \( v_i \) and \( v_j \) in \( \sigma \).

The next lemma generalizes Lemma 3 by taking into account a set \( E' \) of disjoint edges.

**Lemma 4** Let \( G \) be an outerplanar graph with \( n \) vertices and let \( E' \subseteq E(G) \) be a set of disjoint edges. There exists an \( O(n) \)-time algorithm that adds edges to \( G \) in such a way that the augmented graph \( G' \) has a 2-page book embedding \( \phi(G') \) and has a Hamiltonian cycle containing all edges of \( E' \). The 2-page book embedding \( \phi(G') \) has the same linear ordering \( \sigma \) as \( \phi(G) \) and all the edges of \( E'(G') \cap E(G) \) are in the same page of \( \phi(G') \).

**Proof:** Since \( G \) is outerplanar it admits a 1-page book embedding [1]. We first prove that it is possible to find a 1-page book embedding \( \phi(G) \) of \( G \) such that the first and the last vertex in the linear ordering of \( \phi(G) \) are not connected by an edge of \( E' \). Let \( \phi(G) \) be any 1-page book embedding of \( G \) with linear ordering \( \sigma' \); if the first and the last vertex of \( \sigma' \) are not connected by an edge of \( E' \) we are done. Otherwise, since the edges in \( E' \) do not form a cycle, there must exist a pair of vertices \( u \) and \( v \) consecutive in \( \sigma' \) that are not connected by an edge of \( E' \). By repeatedly applying Lemma 2, we can compute a 1-page book embedding \( \phi(G) \) of \( G \) such that \( u \) and \( v \) are the first and the last vertex in the linear ordering of \( \phi(G) \), respectively.

Let \( v_0, v_1, \ldots, v_{n-1} \) be the total ordering \( \sigma \) of \( \phi(G) \). Let \( e_0 = (v_0, v_1), e_1 = (v_1, v_2), \ldots, e_{n-1} = (v_{n-2}, v_{n-1}) \) be the edges of \( E' \) not nested inside any other edge of \( E' \). Since the edges of \( E' \) are disjoint then \( i_0 < j_0 < i_1 < j_1 < \cdots < i_{h-1} < j_{h-1} \). Also, by Lemma 3 for each edge \( e_l \) \((0 \leq l \leq h-1)\) it is possible to add edges to \( G \) so that in the augmented graph there exists a simple path \( \pi_l \) from \( v_{i_l-1} \) to \( v_{j_l+1} \) that contains edge \( e_l \), all edges of \( E' \) nested inside \( e_l \), and all vertices between \( v_{i_l} \) and \( v_{j_l+1} \). We choose a Hamiltonian cycle \( C = v_0, v_1, \ldots, v_0, v_1, \ldots, v_{h-1}, v_{h-2}, v_{h-1}, v_0 \), where the edges of \( C \) that are not in \( G \) are augmenting edges. Let \( G' \) be the augmented graph obtained by the edge addition described above. We partition the edges of \( G' \) into two pages: \( E_0 \) contains the edges of \( E(G') \cap E(G) \), and \( E_1 \) contains the edges of \( E(G') \setminus E(G) \). We prove that the linear ordering \( \sigma \) (which is a linear ordering of \( G' \) since \( V(G') = V(G) \)) along with the partition of \( E(G') \) into the two pages \( E_0 \) and \( E_1 \) is a 2-page book embedding.

The edges in \( E_0 \) do not cross each other because they are all edges of \( G \) and do not cross in \( \phi(G) \). Let \( d_0 = (v_{i_0}, v_{h_0}) \) and \( d_1 = (v_{j_1}, v_{h_1}) \) be two edges in \( E_1 \) both connecting two vertices that are non-consecutive in \( \sigma \). If \( d_0 \) and \( d_1 \) are both edges of a simple path \( \pi_l \) \((0 \leq l \leq h-1)\), then they do not cross by Lemma 3. If \( d_0 \) and \( d_1 \) are edges of two different simple paths \( \pi_{i_0} \) and \( \pi_{i_1} \) \((0 \leq i_0 < i_1 \leq h \leq h-1)\), then, by Lemma 3, we have that \( i_0 < g_0 < h_0 < j_{i_0} + 1 \) and \( i_1 < g_1 < h_1 < j_{i_1} + 1 \). Since \( j_{i_0} + 1 \leq i_1 \), then \( d_0 \) and \( d_1 \) do not cross. The time complexity of the algorithm described above is \( O(n) \) since each path \( \pi_l \) \((0 \leq l \leq h-1)\) can be computed in \( O(|j_l - i_l|) \) by Lemma 3 and \( C \) can be computed by visiting the remaining vertices of \( G \) in the order defined by \( \sigma \).

We are now ready to further generalize Lemma 3 and consider a set \( E' \) of disjoint paths. To this aim we first define a suitable subgraph of \( G \) and show how to compute a 1-page book embedding for such graph. Let \( G \) be an outerplanar graph and let \( E' \subseteq E(G) \) be a set of edges of \( G \) such that the edges of \( E' \) form a forest \( \mathcal{F} \) of simple paths. The *simplified graph of \( G \) with respect to \( E' \)* is a graph \( H \) defined as follows. The vertices of \( H \) are the same as \( G \), except those having degree two in some path of \( \mathcal{F} \). There is an edge \((v_i, v_j)\) in \( H \) if and only if there exists a simple path in \( \mathcal{F} \) that has endvertices \( v_i \) and \( v_j \). Figure 6(a) shows an outerplanar graph \( G \). A set \( E' \) of edges of \( G \) are highlighted. The simplified graph \( H \) of \( G \) with respect to \( E' \) is shown in Figure 6(b); note that vertex \( v \) of Figure 6(a) is not in the simplified graph because it has two adjacent edges of \( E' \) in \( G \).

**Lemma 5** Let \( G \) be an outerplanar graph with \( n \) vertices and let \( E' \subseteq E(G) \) be a set of edges of \( G \) such that the edges of \( E' \) form a forest of simple paths. Let \( H \) be the simplified graph of \( G \) and let \( \phi(G) \) be a 1-page book embedding of \( G \) with linear ordering \( \sigma_G \). Then the linear ordering \( \sigma_H \) of \( H \) induced by \( \sigma_G \) defines a 1-page book embedding \( \phi(H) \) of \( H \). Moreover, \( H \) and \( \phi(H) \) can be computed from \( G \) and \( \phi(G) \) in \( O(n) \) time.

**Proof:** Let \( \mathcal{F} \) be the forest of simple paths formed by the edges of \( E' \). Let \( e_0 = (u_0, v_0) \) and \( e_1 = (u_1, v_1) \) be two edges of \( H \). Assume as a contradiction that \( e_0 \) and \( e_1 \) cross, i.e. \( u_0 <_{\sigma_H} u_1 <_{\sigma_H} v_0 <_{\sigma_H} v_1 \). Since
σ_H is induced by σ_G we also have u_0 <_σ_0 u_1 <_σ_1 v_0 <_σ_1 v_1. Let π_0 and π_1 be the simple paths of F that correspond to e_0 and e_1 respectively. Since u_0 <_σ_0 u_1 <_σ_1 v_0 and π_0 connects u_0 to v_0, there exists in φ(G) an edge e_2 = (u_2, v_2) ∈ π_0 such that u_2 <_σ_0 u_1 <_σ_0 v_2. Two cases are possible.

Case 1: v_2 <_σ_2 v_1. Since u_1 is between u_2 and v_2 in σ_G, v_1 is after v_2 in σ_G, and π_1 connects u_1 to v_1, there exists in φ(G) an edge e_3 = (u_3, v_3) ∈ π_1 such that either u_2 <_σ_3 u_3 <_σ_3 v_2 <_σ_3 v_3 (see Figure 7(a)) or u_3 <_σ_2 u_2 <_σ_2 v_3 <_σ_2 v_2 (see Figure 7(b)). In both cases e_2 and e_3 cross in φ(G), which is impossible.

Case 2: v_1 <_σ_1 v_2. Since v_0 is between u_1 and v_1 in σ_G and π_1 connects u_1 to v_1, there exists in φ(G) an edge e_3 = (u_3, v_3) ∈ π_1 such that u_3 <_σ_0 v_0 <_σ_0 v_3. Since v_0 is between u_3 and v_3 in σ_G and π_0 connects v_0 to v_2, there exists in φ(G) an edge e_4 = (u_4, v_4) ∈ π_0 such that either u_3 <_σ_0 u_4 <_σ_0 v_3 <_σ_0 v_4 (see Figure 8(a)) or u_4 <_σ_2 u_3 <_σ_2 v_4 <_σ_2 v_3 (see Figure 8(b)). In both cases e_3 and e_4 cross in φ(G), which is impossible.

The simplified graph H of G and a 1-page book embedding φ(H) of H can be computed in O(n) time by finding the maximal paths of G consisting of edges of E(G) ∩ E(P).

A 1-page book embedding φ(G) of the outerplanar graph of Figure 6(a) is depicted in Figure 9(a). The 1-page book embedding φ(H) of the simplified form H of G with respect to the highlighted edges is shown in Figure 9(b). The simplified graph H and its book embedding are used to prove the following.

Lemma 6 Let G be an outerplanar graph with n vertices and let E' ⊂ E(G) be a set of edges of G that form a forest of simple paths. There exists an O(n)-time algorithm that adds edges to G in such a way that the augmented graph G' is planar and has a Hamiltonian cycle containing all edges of E'.
Proof: Let \( F \) be the forest of simple paths formed by the edges of \( E' \) and let \( H \) be the simplified graph of \( G \) with respect to \( E' \). By Lemma 5, the linear ordering \( \sigma_H \) induced by \( \sigma_G \) defines a 1-page book embedding \( \phi(H) \) of \( H \). Since \( H \) is outerplanar and since all edges of \( H \) are disjoint by definition, we can apply Lemma 4 to \( H \) where the set of disjoint edge is \( E(H) \) itself. We obtain an augmented graph \( H' \) that has a 2-page book embedding \( \phi(H') \) and has a Hamiltonian cycle \( D \) containing all edges of \( H' \) (actually, cycle \( D \) coincides with \( H' \) since \( E' \) coincides with \( E(H) \)). The 2-page book embedding \( \phi(H') \) obtained by augmenting the 1-page book embedding \( \phi(H) \) of Figure 9(b) is shown in Figure 10(a).

Consider now the graph \( G' \) constructed starting from \( G \) by adding to \( G \) the edges of \( E(H') \setminus E(H) \) that are not yet in \( G \). The cycle obtained by replacing in \( D \) each edge of \( E(H) \) with the corresponding path of \( F \), is a Hamiltonian cycle \( C \) of \( G' \). Indeed, \( C \) is a simple cycle since we replaced a set of disjoint edges of \( D \) (i.e. the edges of \( E(H) \)) with a set of disjoint paths (i.e. the paths of \( F \)) such that: (i) each path has the same endvertices of the corresponding edge; (ii) each path does not share any vertex with \( D \) except its endvertices. Also, \( C \) contains all the vertices of \( G' \) since it contains all the vertices of \( H \) and all the vertices of the paths of \( F \). Figure 10(b) shows a 2-page book embedding \( \phi(G') \) of the augmented graph \( G' \) of the graph \( G \) of Figure 10(c). Figure 9(a) shows a different representation of \( G' \). In both pictures the edges of the Hamiltonian cycle \( C \) of \( G' \) are highlighted.

We now prove that \( G' \) is planar. The edges of \( G' \) are partitioned into two pages: page \( E_0 \) contains the edges of \( E(G') \cap E(G) \), and page \( E_1 \) contains the edges of \( E(G') \cap E(H') \). We prove that the linear ordering \( \sigma_G \) (that is a linear ordering of \( G' \) since \( V(G') = V(G) \)) along with the partition of \( E(G') \) into the two pages \( E_0 \) and \( E_1 \) is a 2-page book embedding of \( G' \).

The edges of \( E_0 \) do not cross each other because they are edges of \( G \) and do not cross in \( \phi(G) \). Concerning the edges of \( E_1 \), we have that by Lemma 4 the linear ordering of \( \phi(H') \) is the same as the linear ordering
Figure 9: (a) A 1-page book embedding of the graph $G$ of Figure 6(a). (b) A 1-page book embedding of the graph $H$ of Figure 6(b).

of $\phi(H)$, i.e. it coincides with the linear ordering of $H'$ induced by $\sigma_G$. Also the edges of $E(H') \setminus E(H)$ are in page $E_1$ of $\phi(H')$. As a consequence, if two edges of page $E_1$ of $\phi(G)$ (i.e. two edges of $E(G') \cap E(H')$) cross, then the same two edges cross in the page $E_1$ of $\phi(H')$, which is impossible because $\phi(H')$ is a 2-page book embedding by Lemma 4. Since $G'$ admits a 2-page book embedding it is sub-Hamiltonian and hence planar.

By Lemma 5 $H$ and $\phi(H)$ can be computed in $O(n)$ time from $G$ and $\phi(G)$. By Lemma 4, $H'$ and $\phi(H')$ can be computed in $O(n)$ time from $H$ and $\phi(H)$. Finally, $G'$ and $C$ can be computed in $O(n)$ time from $H'$ and $D$ by replacing each edge of $E(H)$ with the corresponding path of $F$ and by adding to $G$ the edges of $E(H') \setminus E(H)$.

We are now ready to prove the main result of this subsection.

**Theorem 4** Let $G$ be an outerplanar graph and let $P$ be a simple path such that $V(G) = V(P) = V$. $G$ and $P$ can be simultaneously embedded with fixed edges in $O(n)$ time, using at most one bend for each edge of $G$ and zero bends for each edge of $P$, on an integer grid of size $O(n) \times O(n^2)$, where $n = |V|$.

**Proof:** A simultaneous embedding with fixed edges of $G$ and $P$ can be computed as follows. Path $P$ is augmented to become a cycle $C$ by adding an edge connecting its endvertices. The edges of $E(G) \cap E(P)$ form a forest of paths. By Lemma 6 we can find an (augmented) Hamiltonian cycle $C$ of $G$ that contains all the edges of $E(G) \cap E(P)$. $G$ and $P$ are then simultaneously embedded by means of Lemma 1. The fact that we have a simultaneous embedding with fixed edges follows from Observation 1. The bound on the area follows from Observation 2.

Concerning the time complexity we observe that a 1-page book-embedding $\phi(G)$ of an outerplanar graph $G$ can be computed in $O(n)$ time. Namely, given any outerplanar embedding of $G$ (that can be computed in $O(n)$ time), the linear ordering obtained by choosing an arbitrary vertex $v$ of $G$ and traversing the boundary of the external face of $G$ defines a 1-page book embedding of $G$. By Lemma 6 we can find a Hamiltonian cycle $C$ of $G$ that contains all the edges of $E(G) \cap E(P)$ in $O(n)$ time. By Lemma 1 the simultaneous embedding of $G$ and $P$ can be computed in $O(n)$ time.

### 4.2 Extensions and Generalizations

The result of Theorem 4 can be generalized to the case in which the sets of vertices of the two graphs involved in the simultaneous embedding do not coincide but there is only a subset of their vertices in common. In
Figure 10: (a) A 2-page book embedding $\phi(H')$ obtained by augmenting the 1-page book embedding $\phi(H)$ of Figure 9(b) according to the technique described in the proof of Lemma 6. (b) A 2-page book embedding $\phi(G')$ obtained by augmenting the 1-page book embedding $\phi(G)$ of Figure 9(b) according to the technique described in the proof of Lemma 6. The Hamiltonian cycle $C$ of $G'$ is highlighted. (c) The augmented graph $G'$. The Hamiltonian cycle $C$ of $G'$ is highlighted.
this case, the definition of simultaneous embedding with fixed edges given in Section 2 can be extended as follows. Let $G_1$ and $G_2$ be a pair of planar graphs such that $V(G_1) \cap V(G_2) \neq \emptyset$. A simultaneous embedding with fixed edges of $G_1$ and of $G_2$ is a pair of crossing-free drawings $\Gamma_1$ and $\Gamma_2$ of $G_1$ and $G_2$, respectively, such that for every vertex $v \in V(G_1) \cap V(G_2)$ we have $\Gamma_1(v) = \Gamma_2(v)$ and for every edge $e \in E(G_1) \cap E(G_2)$ we have $\Gamma_1(e) = \Gamma_2(e)$. We start by extending the technique of Brass et al. [2] concerning simultaneous geometric embedding of two simple paths.

**Lemma 7** Let $P_1$ and $P_2$ be two simple paths having some of their vertices in common. $P_1$ and $P_2$ admit a simultaneous geometric embedding on an integer grid of size $n_1 \times n_2$, where $n_1 = |V(P_1)|$ and $n_2 = |V(P_2)|$. The simultaneous embedding can be computed in $O(n_1 + n_2)$ time.

**Proof:** The proof extends the ideas of Brass et al. [2]. Starting from one of its end-vertices, $P_1$ is traversed and an increasing positive integer is given to each visited vertex; for each vertex $v \in V(P_1)$, this number is the $x$-coordinate of the point representing $v$. Starting from one of its end-vertices, $P_2$ is traversed and a second increasing positive integer is given to each visited vertex; for each vertex $v \in V(P_2)$, this number is the $y$-coordinate of the point representing $v$. At the end of this procedure, each vertex of $V(P_1) \cap V(P_2)$ is given the $x$- and $y$-coordinate of the point representing it. The $y$-coordinate of a vertex $v$ in $V(P_1) \setminus V(P_2)$ is any arbitrarily chosen number in the interval $[0, n_2 - 1]$. The $x$-coordinate of a vertex $v$ in $V(P_2) \setminus V(P_1)$ is any arbitrarily chosen number in the interval $[0, n_1 - 1]$. The defined set of points supports a geometric simultaneous embedding of $P_1$ and of $P_2$ because each of the two drawings is either $x$- or $y$-monotone and hence it is crossing-free. □

By means of Lemma 7 we can prove the following extension of Theorem 4.

**Theorem 5** Let $G$ be an outerplanar graph and let $P$ be a simple path such that $V(G) \cap V(P) \neq \emptyset$. $G$ and $P$ can be simultaneously embedded with fixed edges in $O(n)$ time, using at most one bend for each edge of $G$ and zero bends for each edge of $P$, on an integer grid of size $O(n) \times O(n^2)$, where $n = |V(G) \cup V(P)|$.

**Proof:** As in the proof of Theorem 4 we can augment $G$ and $P$ to find a Hamiltonian cycle $C_1$ of $G$ and a Hamiltonian cycle $C_2$ of $P$ such that the two cycles contain all edges of $E(G) \cap E(P)$. A simultaneous embedding of $G$ and $P$ can now be computed in three steps, by a technique analogous to that of Section 3. Namely, in Step 1 two Hamiltonian paths $P_1$ and $P_2$ are obtained by removing a closing edge from $C_1$ and from $C_2$. In Step 2 a simultaneous geometric embedding of $P_1$ and $P_2$ is computed by the technique described in the proof of Lemma 7. In Step 3, the remaining edges of $G$ are added by the technique of Kaufmann and Wiese [15], which produces a crossing-free drawing of $G$. Since the edges of $E(G) \cap E(P)$ are straight-line segments, we have a simultaneous embedding with fixed edges.

The bounds on the area and on the time complexity can be proved with the same reasoning of Section 3. □

One of the intriguing problems left as open in [11] is whether or not it is possible to simultaneously embed two trees with fixed edges. The following lemma studies simultaneous embedding with fixed edges of two outerplanar graphs and provides a partial answer to the open problem.

**Theorem 6** Let $G_1$ and $G_2$ be two outerplanar graphs such that $V(G_1) = V(G_2) \neq \emptyset$. If the edges of $E(G_1) \cap E(G_2)$ form a forest of paths, then $G_1$ and $G_2$ can be simultaneously embedded with fixed edges in $O(n)$ time, using at most one bend per edge, on an integer grid of size $O(n^3) \times O(n^2)$, where $n = |V(G_1) \cup V(G_2)|$.

**Proof:** Since the subgraph induced by $E(G_1) \cap E(G_2)$ is a forest of paths, we can apply Lemma 6 to both $G_1$ and $G_2$ and find two (augmented) Hamiltonian cycles $C_1$ of $G_1$ and $C_2$ of $G_2$ such that the edges of $E(G_1) \cap E(G_2)$ belong to both $C_1$ and $C_2$. A simultaneous embedding of $G_1$ and $G_2$ can now be computed in three steps, by a technique analogous to that of Section 3. Namely, in Step 1 two Hamiltonian paths $P_1$ and $P_2$ are obtained by removing a closing edge from $C_1$ and from $C_2$. In Step 2 a simultaneous geometric embedding of $P_1$ and $P_2$ is computed by the technique described in the proof of Lemma 7. In Step 3, the remaining edges of $G_1$ and $G_2$ are added by the technique of Kaufmann and Wiese [15], which produces two
crossing-free drawings of $G_1$ and of $G_2$. Since the edges of $E(G) \cap E(P)$ are straight-line segments, we have a simultaneous embedding with fixed edges.

The bounds on the area and on the time complexity can be proved with the same reasoning of Section 3.

A further extension of Theorem 5 is about simultaneous embedding of an outerplanar graph and a cycle.

**Theorem 7** Let $G$ be an outerplanar graph and let $C$ be a simple cycle such that $V(G) \cap V(C) \neq \emptyset$. $G$ and $C$ can be simultaneously embedded with fixed edges in $O(n)$ time, using at most one bend per edge, on an integer grid of size $O(n^2) \times O(n^2)$, where $n = |V(G) \cup V(C)|$.

**Proof:** If all edges of $C$ are also edges of $G$, it suffices to compute a drawing of the outerplanar graph $G$. For example choose $n = |V(G)|$ collinear points and use the technique by Kaufmann and Wiese [15] to obtain the bound on the time complexity, number of bends per edge, and grid size.

If otherwise there exists an edge $e$ of $C$ such that $e$ is not an edge of $G$, then we can apply Theorem 6 to $G$ and $C$ and the statement holds.

5 Conclusion and Open Problems

In this paper we have presented an algorithm that computes a simultaneous embedding of an outerplanar graph and a path with fixed edges on an integer grid of size $O(n) \times O(n^2)$. We have then extended the algorithm and we have proved that an outerplanar graph and a cycle can be simultaneously embedded with fixed edges on an integer grid of size $O(n^2) \times O(n^2)$ and that a pair of outerplanar graphs can be simultaneously embedded with fixed edges on an integer grid of size $O(n^2) \times O(n^2)$ if they share a set of edges that form a forest of paths. Our techniques exploit and extend those presented in [11].

Several questions concerning simultaneous embeddings remain open. We list here those that, in our opinion, are the most interesting.

- Theorem 6 gives a result about simultaneous embedding with fixed edges of two outerplanar graphs in the special case when the two graphs share a forest of paths. Is it possible to simultaneously embed with fixed edges two general outerplanar graphs?
- The problem above can be stated in more general terms. It is possible to simultaneously embed with fixed edges two planar graphs?
- Concerning the simultaneous embedding (without fixed edges) of two planar graphs, is it possible to reduce the number of bends from two to one?

References


